Rutgers University: Algebra Written Qualifying Exam
August 2014: Problem 3 Solution

Exercise. Let S be an integral domain and let $a \in S$. Let R be a subring of S such that $S = R[a]$. Prove or disprove the following:

(a) If R is a principal ideal domain, then S is a principal ideal domain.

Solution.

False: $\mathbb{Z}[x]$ is an integral domain and $x \in \mathbb{Z}[x]$
$\mathbb{Z}$ is a PID but $\mathbb{Z}[x]$ is not a PID.

Proof that $\mathbb{Z}$ is a PID:
Let $I \subseteq \mathbb{Z}$ be an ideal of $\mathbb{Z}$ and $m \in I$ be the smallest positive integer in I
If $a \in I$ then

$$\begin{aligned} & a = bm + r && 0 \leq r < b \\ \implies & a - bm = r \in I && \\ \implies & r = 0 && \text{by choice of } m \\ \implies & I = \langle m \rangle && \end{aligned}$$

Prove that $\mathbb{Z}[x]$ is not a PID:

$$\langle 2, x \rangle = \{2s + tx : s, t \in \mathbb{Z}[x]\} \qquad \text{is an ideal of } \mathbb{Z}[x]$$

If $\exists q \in \mathbb{Z}[x]$ s.t. $\langle q \rangle = \langle 2, x \rangle$ then $\exists r(x), s(x) \in \mathbb{Z}[x]$ such that

$$\begin{aligned} & q(x)r(x) = 2 \quad \text{and} \quad q(x)s(x) = x \\ & q(x)r(x) = 2 \implies q(x) = 1 \quad \text{or} \quad q(x) = 2 \\ & \text{And } q(x)s(x) = 1 \implies q(x) = 1 \quad \text{or} \quad q(x) = x \\ \implies & q(x) = 1 \\ \text{BUT} \quad & \langle 2, x \rangle \neq \langle 1 \rangle \quad \text{since} \quad 1 \notin \langle 2, x \rangle \end{aligned}$$

Thus, $\mathbb{Z}[x]$ is not a PID

(b) If R is noetherian, then S is noetherian. You may use major theorems in your justification as long as they are specifically mentioned.

Solution.

A **Noetherian ring** is a ring that sarsfies the ascending cahin on left and right ideals.
i.e. given any chain of left (or right) ideals

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{k-1} \subseteq I_k \subseteq I_{k+1} \subseteq \dots$$

$\exists n$ such that $I_n = I_{n+1} = \dots$

This is true by **Hilbert's Basis Theorem**